Introduction to Majorana fermions: part I

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References

• Main references:

• Theoretical references:
References

• Experimental references:

• Reading materials:
Trivia about Majorana (fermion)

• About Ettore Majorana
  – Aug. 5, 1906: born in Catania, Sicily
  – March 1938: disappeared in Palermo, Sicily

• About Majorana fermion
  – Real solutions of Dirac equation, are their own anti-particles.
  – Elementary particles: Neutrinos?
  – 2D condensed matter systems: ν=5/2 FQHE state? Sr$_2$RuO$_4$? ...
  – 1D condensed matter systems: semiconductor nanowire?
§1: Basics about Majorana fermion
Basics of Majorana fermion: preview

• Majorana fermions are “real-valued” fermion modes
  – “real-valued”: they are their own anti-particles: in contrast to “complex” fermions: e.g. electrons vs. positrons.
  – “fermion”: different Majoranas anti-commute.
  – Majorana fermions may obey non-Abelian statistics: Ising anyons.
    • might be used for quantum computation.
    • c.f. Nayak RMP'08
Defining Majorana fermion

- Consider one fermion mode: $\psi$
  - 2d Hilbert space $\mathcal{H}$ spanned by unoccupied and occupied states:
    $$|0\rangle \quad |\psi\rangle$$
  - Fermion creation/annihilation operators
    $$\hat{\psi}^\dagger = |\psi\rangle\langle 0| = (|0\rangle, |\psi\rangle) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \langle 0| \\ \langle \psi| \end{pmatrix}$$
    $$\hat{\psi} = |0\rangle\langle \psi| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
  - Anti-commutation relation:
    $$\{\hat{\psi}, \hat{\psi}^\dagger\} = 1 \quad \{\hat{\psi}, \hat{\psi}\} = \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = 0$$
Defining Majorana fermion

- Majorana fermion operators from one fermion mode:

\[
\hat{\sigma}_1 \equiv \hat{\psi} + \hat{\psi}^\dagger = |\psi\rangle \langle 0| + |0\rangle \langle \psi| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
\hat{\sigma}_2 \equiv i(\hat{\psi}^\dagger - \hat{\psi}) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\]

- “Real-valuedness”: \(\hat{\sigma}_i^\dagger = \hat{\sigma}_i\)

- Anti-commutation relation: \(\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij}\) esp. \(\{\hat{\sigma}_1, \hat{\sigma}_2\} = 0\)

- Fermion number: \(\hat{n} \equiv \hat{\psi}^\dagger \hat{\psi} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1 - i \hat{\sigma}_2 \hat{\sigma}_1}{2}\)

- Fermion number parity:

\[
\hat{P} = i \hat{\sigma}_2 \hat{\sigma}_1 = (-1)^\hat{n} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to \begin{cases} +1 & \text{for } |0\rangle, \\
-1 & \text{for } |\psi\rangle. \end{cases}
\]

\[\hat{\sigma}_{1(2)} \hat{P} = -\hat{P} \hat{\sigma}_{1(2)}\]
Defining Majorana fermion

• Two fermion modes: $\psi_1, \psi_2$
  
  – 4d Hilbert space: tensor product of two 2d Hilbert space.
    \[ \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \]
  
  – Basis: tensor products of single fermion basis
    \[
    |0\rangle = |0\rangle_1 \otimes |0\rangle_2 \equiv |0, 0\rangle \\
    |\psi_2\rangle = |0\rangle_1 \otimes |\psi_2\rangle = \hat{\psi}_2^\dagger |0\rangle \equiv |0, 1\rangle \\
    |\psi_1\rangle = |\psi_1\rangle_1 \otimes |0\rangle_2 = \hat{\psi}_1^\dagger |0\rangle \equiv |1, 0\rangle \\
    |\psi_1 \psi_2\rangle = |\psi_1\rangle_2 \otimes |\psi_2\rangle = \hat{\psi}_1^\dagger \hat{\psi}_2^\dagger |0\rangle = -\hat{\psi}_2^\dagger \hat{\psi}_1^\dagger |0\rangle \equiv |1, 1\rangle
    \]

  – Anti-commutation relation:
    \[
    \{\hat{\psi}_i, \hat{\psi}_j^\dagger\} = \delta_{ij} \quad \{\hat{\psi}_i, \hat{\psi}_j\} = \{\hat{\psi}_i^\dagger, \hat{\psi}_j^\dagger\} = 0
    \]
Defining Majorana fermion

- Two fermion modes: $\psi_1$, $\psi_2$ (cont'd)

\[
\hat{\psi}_1 = |0\rangle\langle \psi_1 | + |\psi_2\rangle\langle \psi_1 \psi_2 | \\
\hat{\psi}_2 = |0\rangle\langle \psi_2 | - |\psi_1\rangle\langle \psi_1 \psi_2 |
\]

\[
\hat{\psi}_1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} \otimes \sigma_0
\]

\[
\hat{\psi}_2 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & -1 \\
0 & 0 \\
0 & 0
\end{pmatrix} \otimes \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix} = \sigma_3 \otimes \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]
Defining Majorana fermion

- **Majorana fermions from two fermion modes:**

\[
\hat{\gamma}_1 \equiv \hat{\psi}_1 + \hat{\psi}_1^\dagger = \sigma_1 \otimes \sigma_0 \quad \hat{\gamma}_2 \equiv i(\hat{\psi}_1 - \hat{\psi}_1^\dagger) = \sigma_2 \otimes \sigma_0 \\
\hat{\gamma}_3 \equiv \hat{\psi}_2 + \hat{\psi}_2^\dagger = \sigma_3 \otimes \sigma_1 \quad \hat{\gamma}_4 \equiv i(\hat{\psi}_2^\dagger - \hat{\psi}_2) = \sigma_3 \otimes \sigma_2
\]

- "Real-valuedness": \( \hat{\gamma}_i^\dagger = \hat{\gamma}_i \)

- Anti-commutation relation: \( \{\hat{\gamma}_i, \hat{\gamma}_j\} = 2\delta_{ij} \)

- Fermion number:

\[
\hat{n}_1 \equiv \hat{\psi}_1^\dagger \hat{\psi}_1 = (1 - i \hat{\gamma}_2 \hat{\gamma}_1)/2 \\
\hat{n}_2 \equiv \hat{\psi}_2^\dagger \hat{\psi}_2 = (1 - i \hat{\gamma}_4 \hat{\gamma}_3)/2
\]

- Fermion number parity:

\[
\hat{P} = i \hat{\gamma}_4 \hat{\gamma}_3 \cdot i \hat{\gamma}_2 \hat{\gamma}_1 = (-1)^{\hat{n}_2 + \hat{n}_1} = \begin{cases} 
+1, & \hat{n}_1 + \hat{n}_2 \text{ is even,} \\
-1, & \hat{n}_1 + \hat{n}_2 \text{ is odd.}
\end{cases}
\]

\[\hat{\gamma}_i \hat{P} = -\hat{P} \hat{\gamma}_i\]
Defining Majorana fermion

- $N$ fermion modes: $\psi_1, \cdots, \psi_N$
  - $2^N$-dim'l Hilbert space: $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$
  - basis: $|n_1, n_2, \ldots, n_N\rangle$, $n_i = 0, 1$

- $2N$ Majorana fermions: c.f. Jordan-Wigner transformation
  \[
  \gamma_{2i-1} = \hat{\psi}_i + \hat{\psi}_i^\dagger = \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0
  \]
  \[
  \gamma_{2i} = i(\hat{\psi}_i^\dagger - \hat{\psi}_i) = \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0
  \]
  - Fermions from Majoranas: $\hat{\psi}_i = (\gamma_{2i-1} + i\gamma_{2i})/2$
  - Fermion number parity:
  \[
  \hat{P} = i\gamma_{2N}\gamma_{2N-1} \cdots i\gamma_2\gamma_1 = (-1)^{\sum_{i=1}^{N} \hat{n}_i} = \sigma_3 \otimes \cdots \otimes \sigma_3
  \]
  \[
  \gamma_i \hat{P} = -\hat{P} \gamma_i
  \]
Properties of Majorana fermion

- “Real-valuedness”: $\hat{\gamma}_i^\dagger = \hat{\gamma}_i$

- Anti-commutation relation: $\{\hat{\gamma}_i, \hat{\gamma}_j\} = 2\delta_{ij}$
  - Majorana has no “vacuum”: $\hat{\gamma}\hat{\gamma}|\cdot\rangle = |\cdot\rangle \neq 0 \Rightarrow \hat{\gamma}|\cdot\rangle \neq 0$

- “Basis change” of Majorana fermions:
  - Real orthogonal transformation: $\hat{\gamma}_i \rightarrow \sum_j M_{ij} \hat{\gamma}_j$, $M^T M = 1$
  - May/May not change fermion parity: $\prod_i \hat{\gamma}_i \rightarrow \det(M) \prod_i \hat{\gamma}_i$
  - Includes particle-hole transformation of fermions: example, two Majoranas from one fermion

\[
\begin{pmatrix}
\hat{\sigma}_1 \\
\hat{\sigma}_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\hat{\sigma}_1 \\
\hat{\sigma}_2
\end{pmatrix}
\hat{\psi} = \frac{\hat{\sigma}_1 + i\hat{\sigma}_2}{2} \rightarrow \frac{\hat{\sigma}_2 + i\hat{\sigma}_1}{2} = i\hat{\psi}^\dagger
\]
Properties of Majorana fermion

- Non-locality: example
  \[
  \hat{\gamma}_{2i-1} = \hat{\psi}_i + \hat{\psi}_i^\dagger = \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{i-1} \otimes \sigma_1 \otimes \underbrace{\sigma_0 \otimes \cdots \otimes \sigma_0}_{N-i}
  \]
  - \( \hat{\gamma}_{2i-1} \) affects/depends on many sub-Hilbert spaces.
  - Products of odd # of Majoranas have similar property.
  - **NOTE**: Hamiltonian can only contain products of even number of Majorana fermion operators. Examples:
    \[
    \hat{H} = -\mu \hat{\psi}_i^\dagger \hat{\psi}_i = (\mu/2)(i\hat{\gamma}_{2i} \hat{\gamma}_{2i-1} - 1)
    \]
    \[
    \hat{H} = -t(\hat{\psi}_i^\dagger \hat{\psi}_j + \hat{\psi}_j^\dagger \hat{\psi}_i) = (t/2)(i\hat{\gamma}_{2i} \hat{\gamma}_{2j-1} - i\hat{\gamma}_{2i-1} \hat{\gamma}_{2j})
    \]
  - **NOTE**: Hamiltonian preserves fermion parity: \([\hat{\mathcal{P}}, \hat{H}] = 0\)
  - **NOTE**: Hilbert space of \(2N\) Majorana fermions divides into even&odd fermion number sectors, each is of dimension \(2^{N-1}\)
Properties of Majorana fermion

• Non-locality (cont'd):

  – Fermions have the similar property: Hamiltonian cannot contain products of odd # of fermions operators.

  – However there is a *non-trivial bosonic hermitian* operator (observable) from a *single* fermion mode: \( \hat{n} = \hat{\psi}^\dagger \hat{\psi} \)

  – **NOTE**: There is **no** *non-trivial bosonic hermitian* operator from a *single* Majorana: \( \hat{\gamma} \cdot \hat{\gamma} = 1 \)

  – Non-trivial observables must contain two or more Majoranas (information is stored non-locally).
Properties of Majorana fermion

- Non-Abelian statistics: c.f. Nayak et al. RMP'08
  - Abelian statistics: with certain # of fermions at fixed positions,
    - the Hilbert space is 1dim',
    - exchanges of fermion pairs just change the phase of wavefunction. Different fermion pair exchanges commute.

\[
\sigma_{1,2} \quad \phi(r', r, r'', \ldots) \quad \sigma_{2,3} \quad \phi(r', r'', r, \ldots)
\]

\[
\sigma_{2,3} \quad \phi(r, r', r'', \ldots) \quad \sigma_{1,2} \quad \phi(r'', r, r', \ldots)
\]

- Non-Abelian statistics: with \(2N\) Majoranas at fixed positions,
  - the Hilbert space is \(2^N\)-dim',
  - different Majorana pair exchange/braiding do not commute: represented as non-commuting \(2^N \times 2^N\) matrices.
Properties of Majorana fermion

• Non-Abelian statistics (cont'd):
  – Braiding of Majorana fermion: \( \sigma_{i,j} : \hat{\gamma}_i \rightarrow \hat{\gamma}_j, \quad \hat{\gamma}_j \rightarrow -\hat{\gamma}_i \)
  – corresponding unitary transformation on Hilbert space \( \hat{\rho}[\sigma_{i,j}] \)
    satisfies
    \[
    \hat{\rho}[\sigma_{i,j}] \cdot \hat{\gamma}_i \cdot \hat{\rho}[\sigma_{i,j}]^{-1} = \hat{\gamma}_j \\
    \hat{\rho}[\sigma_{i,j}] \cdot \hat{\gamma}_j \cdot \hat{\rho}[\sigma_{i,j}]^{-1} = -\hat{\gamma}_i \\
    \hat{\rho}[\sigma_{i,j}] \cdot \hat{\gamma}_k \cdot \hat{\rho}[\sigma_{i,j}]^{-1} = \hat{\gamma}_k, \quad k \neq i, j
    \]
  – Exercise: check \( \hat{\rho}[\sigma_{i,j}] = (1 - \hat{\gamma}_i \hat{\gamma}_j)/\sqrt{2} \)
  – Non-Abelian statistics: exercise
    \[
    \hat{\rho}[\sigma_{2,3}] \cdot \hat{\rho}[\sigma_{1,2}] = (1/2)(1 - \hat{\gamma}_1 \hat{\gamma}_2 - \hat{\gamma}_1 \hat{\gamma}_3 - \hat{\gamma}_2 \hat{\gamma}_3) \\
    \neq \hat{\rho}[\sigma_{1,2}] \cdot \hat{\rho}[\sigma_{2,3}] = (1/2)(1 - \hat{\gamma}_1 \hat{\gamma}_2 + \hat{\gamma}_1 \hat{\gamma}_3 - \hat{\gamma}_2 \hat{\gamma}_3)
    \]
Summary #1

• Basics of Majorana fermion:
  – “Real-valuedness”: $\hat{\gamma}_i^\dagger = \hat{\gamma}_i$
    • Equal weight superposition of particle and hole.
  – Anti-commutation relation (Clifford algebra): $\{\hat{\gamma}_i, \hat{\gamma}_j\} = 2\delta_{ij}$
    • Majorana has no “vacuum”: $\hat{\gamma}|\text{any state}\rangle \neq 0$
    • Basis changes should be real orthogonal transformations.
    • $i\hat{\gamma}_i\hat{\gamma}_j$ has eigenvalues $\pm 1$ b/c $i\hat{\gamma}_i\hat{\gamma}_j \cdot i\hat{\gamma}_i\hat{\gamma}_j = 1$
  – Non-locality: information is stored in pairs of Majoranas.
  – Non-Abelian statistics.
  – Fermion number parity: $\hat{\mathcal{P}} = (-i)^N \prod_{i=1}^{2N} \hat{\gamma}_i$, $\{\hat{\gamma}_i, \hat{\mathcal{P}}\} = 0$
    • $2^N$-dim'l Hilbert space divides into even&odd subspaces, each is of dimension $2^{N-1}$
§2: Model realization of Majorana fermion
Model realization: the goal

- To realize well-separated localized Majorana zero modes in a system with bulk gap
  - “Majorana zero modes”: \([\hat{\gamma}, \hat{H}] = 0\), \(\gamma\)s do no appear in \(H\). Action of these Majoranas do not change energy.
  - \(2n\) Majorana zero modes: \(2^n\)-fold degenerate ground states. Majorana zero modes act non-trivially in this subspace.
  - Bulk gap: clear separation b/w ground & excited states.
  - Localized and well-separated: local perturbations will not lift the “topologically protected” ground state degeneracy, b/c it cannot involve more than one Majorana mode.

**Energy spectrum**

- bulk excitations
- bulk gap
- \(2^n\) ground states
Model realization: 1D p-wave “superconductor”

• 1D spinless fermion chain with p-wave pairing
  
  
  – $t, \Delta, \mu$ are real parameters. As an example, $N$ is assumed even.

\[
\hat{H} = \hat{H}_0 - \Delta \sum_{i=1}^{N-1} (\hat{\psi}_i \hat{\psi}_{i+1} + \hat{\psi}_{i+1}^\dagger \hat{\psi}_i^\dagger)
\]

\[
\hat{H}_0 = -t \sum_{i=1}^{N-1} (\hat{\psi}_i^\dagger \hat{\psi}_{i+1} + \hat{\psi}_{i+1}^\dagger \hat{\psi}_i) - \mu \sum_{i=1}^{N} (\hat{\psi}_i^\dagger \hat{\psi}_{i+1} - 1/2)
\]
Model realization: 1D p-wave “superconductor”

- Rewrite the Hamiltonian in terms of Majoranas
  \[ \hat{\psi}_i = (\hat{\gamma}_{2i-1} + i\hat{\gamma}_{2i})/2 \]
  - a tight-binding model of Majorana fermions: exercise

  \[ \hat{H} = \frac{i}{2} \left\{ \sum_{i=0}^{N-1} \left[ (\Delta - t)\hat{\gamma}_{2i+1}\hat{\gamma}_{2i} + (\Delta + t)\hat{\gamma}_{2i+2}\hat{\gamma}_{2i-1} \right] \right. \]

  \[ \left. + \mu \sum_{i=0}^{N} \hat{\gamma}_{2i}\hat{\gamma}_{2i-1} \right\} \]
Model realization: 1D p-wave “superconductor”

- Special case #1: trivial phase
  - $t=\Delta=0, \mu<0$: $\hat{H} = \frac{\mu}{2} \sum_{i=1}^{N} i \hat{\gamma}_{2i} \hat{\gamma}_{2i-1} = -\mu \sum_{i=1}^{N} (\hat{\psi}_i^\dagger \hat{\psi}_i - \frac{1}{2})$
  - sum of $N$ mutually commuting terms
  - **Unique** ground state: all $i \hat{\gamma}_{2i} \hat{\gamma}_{2i-1} = +1$ namely $\hat{\psi}_i^\dagger \hat{\psi}_i = 0$
  - Bulk excitations of energy ($-\mu$):
    - one of $i \hat{\gamma}_{2i} \hat{\gamma}_{2i-1} = -1$ namely $\hat{\psi}_i^\dagger \hat{\psi}_i = 1$
Model realization: 1D p-wave “superconductor”

- Special case #2: non-trivial phase
  \[ t = -\Delta > 0, \mu = 0: \quad \hat{H} = -t \sum_{i=1}^{N-1} \mathrm{i} \gamma_{2i+1} \gamma_{2i} \]
  - sum of \( N-1 \) mutually commuting terms \( \mathrm{i} \gamma_{2i+1} \gamma_{2i} \)
  - Ground states: all \( \mathrm{i} \gamma_{2i+1} \gamma_{2i} = +1 \)
  - Bulk excitations of energy \( 2t \): one of \( \mathrm{i} \gamma_{2i+1} \gamma_{2i} = -1 \)
Model realization: 1D p-wave “superconductor”

• Special case #2: non-trivial phase (cont'd)
  - NOTE: \( \hat{\gamma}_1, \hat{\gamma}_{2N} \) do no appear in \( H \).
  - “Majorana zero modes”: \( [\hat{\gamma}_1, \hat{H}] = [\hat{\gamma}_{2N}, \hat{H}] = 0 \)
  - Two-fold degeneracy: \( i\hat{\gamma}_1 \hat{\gamma}_{2N} = \mp 1 \)
  - Action of \( \hat{\gamma}_1(2N) \) switches b/w the two degenerate states.

\[
\begin{array}{cccccc}
\hat{\gamma}_1 & \hat{\gamma}_4 & \hat{\gamma}_5 & \hat{\gamma}_{2N} \\
. & . & . & . \\
\text{---} & \text{---} & \text{---} & \text{---} \\
\text{i=1} & \text{i=2} & \text{i=3} & \text{i=N} \\
-\text{i}t & \text{---} & \text{---} & \text{---} \\
\hat{\gamma}_2 & \hat{\gamma}_3 & \hat{\gamma}_6 & \hat{\gamma}_{2N-1} \\
. & . & . & . \\
\text{---} & \text{---} & \text{---} & \text{---} \\
\end{array}
\]
Model realization: 1D p-wave “superconductor”

• Special case #2: non-trivial phase (cont'd)
  
  – Fermion # parity: \( \hat{\mathcal{P}} = -i\gamma_1\gamma_2N i\gamma_2N-1\gamma_2N-2 \cdots i\gamma_3\gamma_2 \)
  
  – Explicit form of (un-normalized) ground states:

    • Use projectors

    \[
P_i = \frac{1+i\gamma_{2i+1}\gamma_{2i}}{2} = \frac{1+\hat{\psi}_i^\dagger \hat{\psi}_{i+1} + \hat{\psi}_{i+1} \hat{\psi}_i + \hat{\psi}_i^\dagger \hat{\psi}_{i+1}}{2}
    \]

    \[
    = \begin{cases}
    1, & \quad i\gamma_{2i+1}\gamma_{2i} = +1, \\
    0, & \quad i\gamma_{2i+1}\gamma_{2i} = -1.
    \end{cases}
    \]

    \[|\text{even}\rangle = |i\gamma_1\gamma_2N = -1\rangle \propto P_{N-1} \cdots P_1 |0\rangle \]

    \[\propto |000 \ldots \rangle + |110 \ldots \rangle + |101 \ldots \rangle + |011 \ldots \rangle + \ldots \]

    \[|\text{odd}\rangle = |i\gamma_1\gamma_2N = +1\rangle \propto P_{N-1} \cdots P_1 \hat{\psi}_1^\dagger |0\rangle \]

    \[\propto |100 \ldots \rangle + |010 \ldots \rangle + |001 \ldots \rangle + |111 \ldots \rangle + \ldots \]
Model realization: 1D p-wave “superconductor”

- Less-special case #3: non-trivial phase
  - $t > -\Delta > 0$, $\mu = 0$: two Majorana chains with alternating hoppings.
  - The “weak-strong-...-strong-weak” chain have two zero energy edge modes in $N \to \infty$ limit.
    \[
    \hat{\xi}_L \propto \hat{\gamma}_1 + \left( \frac{\Delta + t}{\Delta - t} \right)^{\gamma}_5 + \cdots + \left( \frac{\Delta + t}{\Delta - t} \right)^x \hat{\gamma}_{4x+1} + \cdots
    \]
    \[
    \hat{\xi}_R \propto \hat{\gamma}_{2N} + \left( \frac{\Delta + t}{\Delta - t} \right)^{\gamma}_{2N-4} + \cdots + \left( \frac{\Delta + t}{\Delta - t} \right)^x \hat{\gamma}_{2N-4x} + \cdots
    \]
Model realization: 1D p-wave “superconductor”

- Less-special case #3: non-trivial phase (cont'd)
  - To see \([\hat{\xi}_L, \hat{H}] = 0\), rewrite \(H\) of upper chain as
    \[
    \hat{H} = i \frac{\Delta - t}{2} \sum_x \gamma_{4x} \left( \frac{\Delta + t}{\Delta - t} \gamma_{4x-3} - \gamma_{4x+1} \right) + \ldots
    \]
  - Characteristic length \(\sim 2 / \log |\frac{\Delta - t}{\Delta + t}| \sim \frac{|t|}{|\Delta|} \) \(\sim\) coherent length of pairing, when \(|\Delta| \ll |t|\).

\[
\hat{\xi}_L \propto \gamma_1 + \left( \frac{\Delta + t}{\Delta - t} \right) \gamma_5 + \cdots + \left( \frac{\Delta + t}{\Delta - t} \right)^x \gamma_{4x+1} + \cdots
\]

\[
\begin{align*}
\gamma_1 & \quad \frac{\Delta + t}{2} \quad \gamma_4 \quad \frac{\Delta - t}{2} \quad \gamma_5 \\
\text{amplitude} & \propto 1 \quad 0 \quad \frac{\Delta + t}{\Delta - t}
\end{align*}
\]
Model realization: 1D p-wave “superconductor”

• Generic case: criterion for “non-trivialness”:
  
  – Ref.: Alicea, RepProgPhys'12
  
  – Rewrite $H$ of periodic chain into Bogoliubov-de Gennes form.

\[
\hat{H} = \frac{1}{2} \sum_k \left( \hat{\psi}_k^\dagger, \hat{\psi}_{-k} \right) \left( \epsilon_k \sigma_3 + \Delta_k \sigma_2 \right) \begin{pmatrix} \hat{\psi}_k \\ \hat{\psi}_{-k}^\dagger \end{pmatrix}
\]

\[
\begin{align*}
\epsilon_k &= -\mu - 2t \cos k, \quad \Delta_k = 2\Delta \sin k \\
\text{dispersion} \quad E(k) &= \pm \sqrt{\epsilon_k^2 + \Delta_k^2}
\end{align*}
\]

– under the mapping: $k \mapsto (\epsilon_k, \Delta_k)$ the image of Brillouin zone $0 \leq k < 2\pi$ is a closed loop, winding around the origin odd(non-trivial) or even(trivial) number of times.
Model realization: braiding in 1D

• Ref.: Alicea et al., Nat.Phys. 7, 412 (2011); arXiv:1006.4395
  - move Majorana fermions by gating (tuning local $\mu$)
  - Braiding/exchange in “1D” without “collision” by sidetracks.
Summary #2

- Prototypical model of Majorana zero modes:
  - 1D spinless fermion with p-wave pairing, in the non-trivial "topological superconductor" phase.
  - Majorana zero modes localized on the ends (boundaries between trivial & non-trivial regions).
    Characteristic length ~ coherent length of pairing.
§3: Experimental realization and detection
Experimental realization

- Realization of 1D “spinless” fermion with p-wave pairing:
  - Semiconductor wire with spin-orbit coupling + Zeeman field + proximity to s-wave superconductor: c.f. Oreg PRL'10
Experimental realization

• Realization of 1D “spinless” fermion with p-wave pairing:

• 2D realizations: c.f. Tutorial part 0&II.
Experimental detection

- Zero-bias tunneling conductance: c.f. KTLaw PRL'09
  - Perfect Andreev reflection, conductance = \(2e^2/h\).
  - Exp.: Mourik et al. Science'12; Das et al. NatPhys'12

- Fractional Josephson effect: c.f. Kitaev PhysUsp'01
  - Josephson current vs. flux (phase difference) is \(h/e\)-periodic instead of \(h/2e\).
  - Exp.: Rokhinson et al. NatPhys'12

- Interferometry: c.f. Fu PRL'09
  - For 2D realizations, c.f. Tutorial part 0&II
The End.